On spacelike minimal Lagrangian surfaces in the complex hyperbolic quadric Q_2^*

Department of Mathematics, Graduate School of Science, Hokkaido University
Sihao Zeng *

Abstract

In this talk, we present our results on spacelike minimal Lagrangian surfaces in the two dimensional complex hyperbolic quadric Q_2^*. We will show the equivalence between minimality and flatness of a family of connections and describe the associated isometric $\mathbb{S}^1$-family, and establish a precise correspondence with spacelike maximal surfaces in the anti-de Sitter 3-space $\mathbb{H}_1^3$ through their Gauss maps. By applying loop group method, we construct explicit families including $\mathbb{R}$-equivariant and radially symmetric examples.

1 Introduction

Minimal Lagrangian surfaces in definite Kähler manifolds are now well-established for the ambient space $\mathbb{S}^2 \times \mathbb{S}^2 \cong Q_2$ [3, 8]. In contrast, the theory in indefinite Kähler manifolds is far less developed, where the causal structure introduces phenomena absent in the definite case. In the pseudo-Kähler setting, the causal type of the immersion is part of the problem: one must impose and use the spacelike condition to get a meaningful and well-posed surface theory, where the analytical and geometric behaviors differ significantly from the definite case. This observation naturally leads to the study of spacelike Lagrangian submanifolds.

Since minimal Lagrangian surfaces in symmetric spaces arise as conformal harmonic maps, it is natural to study harmonic maps into symmetric spaces that admit an integrable-systems formulation via families of flat connections on a trivial principal G-bundle. This is realized through the loop group method of J. Dorfmeister, F. Pedit, and H. Wu [4], which applies loop group decompositions of infinite-dimensional Lie groups (the DPW method). From this perspective, spacelike minimal Lagrangian surfaces in symmetric spaces should be amenable to a loop group formulation.

In this talk, we focus on spacelike minimal Lagrangian surfaces in the complex hyperbolic quadric Q_2^*, a non-compact Kähler–Einstein symmetric space isometric to the product of hyperbolic planes $\mathbb{H}^2 \times \mathbb{H}^2$ [5]. Although Q_2^* is the non-compact dual of the complex quadric Q_2 and minimal Lagrangian surfaces in Q_2 have been extensively studied via the loop group method in [6], the transition is not a mere sign change: the underlying geometry, the associated curvature conditions, and the resulting integrable equations differ in essential ways. The primary goal of this paper is to establish a loop group framework for spacelike minimal Lagrangian surfaces in Q_2^* and to demonstrate that this class of surfaces analogous to that in the compact case.

*E-mail: sihao.zeng.f1@elms.hokudai.ac.jp

2 Preliminaries

Definition 2.1. *Let (N, ω) be a Kähler manifold of $\dim_{\mathbb{C}} N = n$ with the Kähler form ω. An immersion $f : M \to N$ from an m-dimensional manifold M into N is said to be totally real if $f^*\omega = 0$. In particular, a totally real immersion f is said to be Lagrangian if $m = n$.*

Let $\mathbb{C}^n_m$ be the complexification of the pseudo-Euclidean space $\mathbb{R}^n_m$ with the complex bilinear form $\langle \, , \, \rangle$ defined by

$$\langle \boldsymbol{z}, \boldsymbol{w} \rangle = -z_1 w_1 - \cdots - z_m w_m + z_{m+1} w_{m+1} + \cdots + z_n w_n, \tag{2.1}$$

where $\boldsymbol{z} = (z_1, \ldots, z_n)$, $\boldsymbol{w} = (w_1, \ldots, w_n) \in \mathbb{C}^n_m$. The standard Hermitian inner product $(\, , \,)$ on $\mathbb{C}^n_m$ is given by $(\boldsymbol{z}, \boldsymbol{w}) = \langle \boldsymbol{z}, \bar{\boldsymbol{w}} \rangle$, where $\bar{\boldsymbol{w}}$ is the conjugate of $\boldsymbol{w}$. The pseudo-hyperbolic space $\mathbb{H}^{2n-1}_{2m-1}(c)$ of dimension $2n-1$, index $2m-1$ and constant sectional curvature $c < 0$ is then

$$\mathbb{H}^{2n-1}_{2m-1}(c) := \left\{ \boldsymbol{z} \in \mathbb{C}^n_m \mid (\boldsymbol{z}, \boldsymbol{z}) = \langle \boldsymbol{z}, \bar{\boldsymbol{z}} \rangle = \frac{1}{c} \right\}.$$

In particular, for $m = 1$, we have the $(2n-1)$-dimensional anti-de Sitter space $\mathbb{H}^{2n-1}_1(c)$. The anti-de Sitter 3-space is denoted by $\mathbb{H}^3_1(-1)$, and the complex anti-de Sitter 3-space is denoted by $\mathbb{CH}^3_1$, and the complex hyperbolic quadric Q^*_2 in $\mathbb{CH}^3_1$ can be realized by

$$Q^*_2 := \left\{ [\boldsymbol{z}] \in \mathbb{CH}^3_1 \mid \langle \boldsymbol{z}, \boldsymbol{z} \rangle = 0 \right\}^0, \tag{2.2}$$

where the superscript 0 denotes a connected component. It is known that Q^*_2 is a homogeneous Kähler-Einstein manifold and it is isometric to $\mathbb{H}^2 \times \mathbb{H}^2$, where the curvatures of the two hyperbolic planes $\mathbb{H}^2$ are normalized to -4, [5, 9].

Let $f : M \to Q^*_2$ be a spacelike Lagrangian conformal immersion from a Riemann surface M into Q^*_2. Moreover, let $\mathbb{D} \subset M$ be a simply connected domain with conformal coordinate $z = x + iy$. Then the induced metric on $\mathbb{D}$ can be computed as

$$ds^2_M = 2e^u \mathrm{d}z \mathrm{d}\bar{z}.$$

Let $\mathfrak{f} : \mathbb{D} \to \mathbb{H}^7_3(-1) \subset \mathbb{C}^4_2$ be a local lift of f, i.e. $f = \pi \circ \mathfrak{f}$, where $\pi : \mathbb{H}^7_3(-1) \to \mathbb{CH}^3_1$ is the Hopf fibration. In fact, the projection f can be realized as $[\mathfrak{f}]$. Since f is conformal and Lagrangian, and satisfies $f(M) \subset Q^*_2$, we obtain

$$\langle \mathfrak{f}_z, \overline{\mathfrak{f}_{\bar{z}}} \rangle = 0, \quad \langle \mathfrak{f}_z, \overline{\mathfrak{f}_z} \rangle = \langle \mathfrak{f}_{\bar{z}}, \overline{\mathfrak{f}_{\bar{z}}} \rangle = e^u, \quad \langle \mathfrak{f}, \mathfrak{f} \rangle = 0, \quad \langle \mathfrak{f}_z, \mathfrak{f} \rangle = \langle \mathfrak{f}_{\bar{z}}, \mathfrak{f} \rangle = 0. \tag{2.3}$$

Here $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ are the complex differentiations.

Definition 2.2. *If a local lift $\mathfrak{f}$ defined above satisfies*

$$\langle \mathfrak{f}_z, \bar{\mathfrak{f}} \rangle = \langle \mathfrak{f}_{\bar{z}}, \bar{\mathfrak{f}} \rangle = 0,$$

then we call $\mathfrak{f}$ a horizontal lift.

Since horizontal lifts of f are not unique, we fix one horizontal lift $\mathfrak{f}$ for time being. Define the indefinite special orthogonal group

$$\mathrm{SO}(2,2) := \left\{ A \in M_{4\times 4}(\mathbb{R}) \mid A^T \eta A = \eta, \ \det A = 1 \right\}, \quad \eta = \mathrm{diag}(-1,-1,1,1).$$

Since the identity component of $\mathrm{SO}(2,2)$ acts transitively on Q^*_2 as the orientation-preserving isometry, Q^*_2 is isomorphic to the symmetric space:

$$Q^*_2 = \mathrm{SO}_0(2,2)/\left(\mathrm{SO}(2) \times \mathrm{SO}(2)\right), \tag{2.4}$$

where the subscript 0 denotes the identity component, see [1, 5]. Indeed, by choosing the involution $\sigma = \mathrm{Ad}\,\mathrm{diag}(1,1,-1,-1)$ on $\mathrm{SO}_0(2,2)$, the fixed point set of σ is exactly $\mathrm{SO}(2)\times\mathrm{SO}(2)$. Let $\mathfrak{f} : M \to Q_2^* = \mathrm{SO}_0(2,2)/\left(\mathrm{SO}(2)\times\mathrm{SO}(2)\right)$ and let $\mathcal{F} : \mathbb{D}\subset M \to \mathrm{SO}_0(2,2)$ be a local lift of f as

$$\mathcal{F} := \left(\frac{1}{\sqrt{2}}\left(\mathfrak{f}+\bar{\mathfrak{f}}\right), -\frac{i}{\sqrt{2}}\left(\mathfrak{f}-\bar{\mathfrak{f}}\right), \frac{\mathfrak{f}_z+\overline{\mathfrak{f}_z}}{\sqrt{2e^u+\alpha+\bar{\alpha}}}, -\frac{i\left\{\mathfrak{f}_z\left(e^u+\bar{\alpha}\right)-\overline{\mathfrak{f}_z}\left(e^u+\alpha\right)\right\}}{\sqrt{\left(2e^u+\alpha+\bar{\alpha}\right)\left(e^{2u}-\alpha\bar{\alpha}\right)}}\right), \tag{2.5}$$

such that $\mathcal{F}(z_0) = \mathrm{Id}$, where $\mathfrak{f}$ is a horizontal lift defined above and

$$\alpha := \langle \mathfrak{f}_z, \mathfrak{f}_z\rangle. \tag{2.6}$$

Its Maurer-Cartan form can be computed as follows:

$$\omega = \mathcal{F}^{-1}\mathrm{d}\mathcal{F} = \mathcal{F}^{-1}\mathcal{F}_z\mathrm{d}z + \mathcal{F}^{-1}\mathcal{F}_{\bar{z}}\mathrm{d}\bar{z}. \tag{2.7}$$

Since Q_2^* is a symmetric space as in (2.4) and minimal surfaces in Q_2^* can be regarded as conformal harmonic maps, thus the integrable systems approach applies. We consider the following family of connection 1-forms $\mathrm{d}+\omega^\lambda$:

$$\omega^\lambda = \lambda^{-1}\omega_{\mathfrak{p}}^{'} + \omega_{\mathfrak{k}} + \lambda\omega_{\mathfrak{p}}^{''}, \quad (\lambda\in\mathbb{S}^1), \tag{2.8}$$

where $\mathfrak{g} = \mathrm{Lie}\left(\mathrm{SO}_0(2,2)\right) = \mathfrak{so}(2,2)$ admits the decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ with the fixed point subalgebra $\mathfrak{k} = \mathrm{Fix}(d\sigma) = \mathfrak{so}(2)\times\mathfrak{so}(2)$ and its complement $\mathfrak{p}$, and $\omega_{\mathfrak{k}}$ and $\omega_{\mathfrak{p}}$ are the $\mathfrak{k}$- and the $\mathfrak{p}$-valued 1-forms. Moreover $'$ and $''$ denote the $(1,0)$- and the $(0,1)$-parts, respectively. While the flatness of $\mathrm{d}+\omega$ corresponds to the flatness of $\left(\mathrm{d}+\omega^\lambda\right)|_{\lambda=1}$, requiring $\mathrm{d}+\omega^\lambda$ to be flat for all $\lambda\in\mathbb{S}^1$ imposes an additional condition of harmonicity on the spacelike Lagrangian surface f.

3 Main Results

The structure equations obtained by a $\mathrm{SO}_0(2,2)$-frame can be expressed in terms of the invariants of $\mathfrak{f}$:

$$e^u := \langle \mathfrak{f}_z, \overline{\mathfrak{f}_z}\rangle, \quad \alpha := \langle \mathfrak{f}_z, \mathfrak{f}_z\rangle, \quad \beta := \langle \mathfrak{f}_z, \mathfrak{f}_{\bar{z}}\rangle \quad \text{and} \quad \phi := e^{-u}\langle \mathfrak{f}_{z\bar{z}}, \overline{\mathfrak{f}_{\bar{z}}}\rangle, \tag{3.1}$$

where $\langle\ ,\ \rangle$ denotes the scalar product in $\mathbb{C}_2^4$ and the subscripts $z, \bar{z}$ are the complex derivatives. Then we obtain the first main result by a straightforward computation.

Theorem 3.1. *Let $f : M \to Q_2^*$ be a spacelike Lagrangian conformal immersion and Φ be the associated one-form defined by $\Phi = \phi\mathrm{d}z$. Then f is minimal if and only if $\Phi = 0$.*

Now introducing a family of connection 1-forms $\mathrm{d}+\omega^\lambda$, parametrized by $\lambda\in\mathbb{S}^1$, such that $\omega^\lambda|_{\lambda=1}$ becomes the Maurer-Cartan form of $\mathfrak{f}$. Then we obtain the second main result by comparing the structure equations.

Theorem 3.2. *Let $f : M \to Q_2^*$ be a spacelike Lagrangian immersion and let $d+\omega^\lambda$ be the family of connections. Then the following statements are equivalent:*

1. *The spacelike Lagrangian immersion f is minimal.*
2. *The connections $d+\omega^\lambda$ are flat for all $\lambda\in\mathbb{S}^1$.*
3. *The quadratic differential $\alpha\,\mathrm{d}z^2$ is holomorphic and $\varphi = \arg(\beta)$ is constant, where α, β are defined in (3.1) and $\arg(\beta)$ denotes the argument of β.*

Consider a new local horizontal lift of a spacelike minimal Lagrangian surface f:

$$\hat{\mathfrak{f}} = e^{-\frac{i\varphi}{2}}\mathfrak{f}, \tag{3.2}$$

where $\varphi = \arg(\beta)$ is constant by Theorem 3.2. The new invariants $\hat{\alpha}$ and $\hat{\beta}$ of $\hat{\mathfrak{f}}$ are given by

$$\hat{\alpha} := \langle \hat{\mathfrak{f}}_z, \hat{\mathfrak{f}}_z \rangle = e^{-i\varphi}\alpha, \quad \hat{\beta} := \langle \hat{\mathfrak{f}}_z, \hat{\mathfrak{f}}_{\bar{z}} \rangle = |\beta|, \tag{3.3}$$

i.e. $|\hat{\alpha}| = |\alpha|$, and $\hat{\beta}$ is a non-negative real function. By $\hat{\beta} = \sqrt{e^{2u} - |\hat{\alpha}|^2}$, all the data in Maurer-Cartan form now can be represented by u and $\hat{\alpha}$.

Definition 3.3. *Denote the new frame of the lift $\hat{\mathfrak{f}}$ in* (3.2) *of a spacelike minimal Lagrangian immersion by $\hat{\mathcal{F}}$. Then by Theorem 3.2, there exists a family of frames $\hat{\mathcal{F}}_\lambda$ such that $\hat{\mathcal{F}}_\lambda|_{\lambda=1} = \hat{\mathcal{F}}$, and we call $\hat{\mathcal{F}}_\lambda$ the extended frame.*

The new family of connection 1-forms $\mathrm{d} + \hat{\omega}^\lambda$ parameterized by $\lambda \in \mathbb{S}^1$ can be explicitly written as follows:

$$\hat{\omega}^\lambda = \hat{\mathcal{F}}_\lambda^{-1}\mathrm{d}\hat{\mathcal{F}}_\lambda. \tag{3.4}$$

Then the flatness condition leads to an *elliptic sinh-Gordon equation*

$$\hat{u}_{z\bar{z}} - e^{\hat{u}} + |\hat{\alpha}|^2 e^{-\hat{u}} = 0, \tag{3.5}$$

where $\hat{u}$ is a real function with

$$2e^u \mathrm{d}z\mathrm{d}\bar{z} = (e^{\hat{u}} + |\hat{\alpha}|^2 e^{-\hat{u}})\mathrm{d}z\mathrm{d}\bar{z}. \tag{3.6}$$

Furthermore, by choosing a suitable gauge transformation, we obtain the third main result.

Theorem 3.4. *Let $f : M \to Q_2^*$ be a spacelike minimal Lagrangian immersion with induced metric $2e^u\mathrm{d}z\mathrm{d}\bar{z}$ and holomorphic quadratic differential $\hat{\alpha}\,\mathrm{d}z^2$. Then there exists an $\mathbb{S}^1$-family of spacelike minimal Lagrangian immersions $\{f^\lambda\}$ with the same induced metric and holomorphic quadratic differential $\hat{\alpha}^\lambda \mathrm{d}z^2 = \lambda^{-2}\hat{\alpha}\,\mathrm{d}z^2$.*

And by the elliptic sinh-Gordon equation (3.5) and the Maurer-Cartan form of $\mathfrak{f}$, we obtain the fourth main result.

Theorem 3.5. *Any spacelike maximal surface f_{max} in $\mathbb{H}_1^3$ with unit normal N, metric $2e^{\hat{u}}\,\mathrm{d}z\mathrm{d}\bar{z}$ and Hopf differential $\mathcal{Q}\,\mathrm{d}z^2$, induces a spacelike minimal Lagrangian surface $f = [f_{max} + iN] \in Q_2^*$, whose holomorphic differential is $-i\mathcal{Q}\mathrm{d}z^2$ and whose metric is*

$$2e^u\,\mathrm{d}z\mathrm{d}\bar{z} = \left(e^{\hat{u}} + |\mathcal{Q}|^2 e^{-\hat{u}}\right)\mathrm{d}z\mathrm{d}\bar{z}. \tag{3.7}$$

Conversely, given a spacelike minimal Lagrangian surface f in Q_2^* with holomorphic differential $\hat{\alpha}\,\mathrm{d}z^2$ and metric $2e^u dz d\bar{z}$, there exists a unique map $g = (f_{max}, N)$ into the timelike unit tangent bundle $T_1^-\mathbb{H}_1^3 = \mathbb{H}_1^3 \times \mathbb{H}^2$ with induced metric $4e^u dz d\bar{z}$, where

$$T_1^-\mathbb{H}_1^3 := \left\{(p, v) \in T\mathbb{H}_1^3 \mid v = (v_0, v_1, v_2) \in T_p\mathbb{H}_1^3 \cong \mathbb{R}_1^3,\ -v_0^2 + v_1^2 + v_2^2 = -1\right\}.$$

Moreover, both projections f_{max} and N have the same Hopf differential $i\hat{\alpha}\,\mathrm{d}z^2$, and the metrics $2e^{\hat{u}} dz d\bar{z}$ of f_{max} and $2e^{\tilde{u}} dz d\bar{z}$ of N are given by

$$e^{\hat{u}} = e^u + \sqrt{e^{2u} - |\hat{\alpha}|^2}, \quad e^{\tilde{u}} = e^u - \sqrt{e^{2u} - |\hat{\alpha}|^2}.$$

Finally, we apply the loop group method to spacelike minimal Lagrangian surfaces in Q_2^* through harmonic maps into $\mathbb{H}^2$ by the well-known Lie group isomorphism

$$\mathrm{SO}_0(2,2) \cong \big(\mathrm{SU}(1,1) \times \mathrm{SU}(1,1)\big)/\boldsymbol{Z}_2,$$

then we construct several examples with this method. We summarize these as the fifth main result: Spacelike minimal Lagrangian surfaces in the complex hyperbolic quadric Q_2^* can be constructed in the following four steps:

1. Solve the initial-value problem:

$$d\Phi = \Phi\xi, \quad \Phi(z_0) = \Phi_0 \in \Lambda\mathrm{SL}(2,\mathbb{C})_\sigma, \tag{3.8}$$

to obtain a unique map $\Phi : \mathbb{D} \to \Lambda\mathrm{SL}(2,\mathbb{C})_\sigma$.

2. Compute the *Iwasawa decomposition* (see [7, 4]) of Φ pointwise on $\mathbb{D}$:

$$\Phi = F_\lambda B, \quad F_\lambda \in \Lambda\mathrm{SU}(1,1)_\sigma, \quad B \in \Lambda^+\mathrm{SL}(2,\mathbb{C})_\sigma, \tag{3.9}$$

Then by [4, Lemma 4.2], F_λ is the extended frame of a harmonic map into $\mathbb{H}^2$. Set the pair of maps given by another map $F_{i\lambda}$ as

$$(F_\lambda, F_{i\lambda}) \in \Lambda\mathrm{SU}(1,1)_\sigma \times \Lambda\mathrm{SU}(1,1)_\sigma. \tag{3.10}$$

3. Use the Loop group isomorphism

$$\Lambda\mathrm{SO}_0(2,2)_\sigma \cong (\Lambda\mathrm{SU}(1,1)_\sigma \times \Lambda\mathrm{SU}(1,1)_\sigma)/\boldsymbol{Z}_2, \tag{3.11}$$

one obtains the extended frame $\mathcal{F}_\lambda \in \Lambda\mathrm{SO}_0(2,2)_\sigma$ of some spacelike minimal Lagrangian immersion into Q_2^*.

4. Finally, by using Proposition 3.6 below, we obtain a family of spacelike minimal Lagrangian immersions f^λ into Q_2^*.

In the following proposition and corollary, we will make use of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proposition 3.6. *Let F_λ be the extended frame defined above. Set*

$$X^\lambda = (X^\lambda_{ij}) := F_\lambda F_{i\lambda}^{-1}, \quad Y^\lambda = (Y^\lambda_{ij}) := iF_\lambda\sigma_3 F_{i\lambda}^{-1}. \tag{3.12}$$

Then the associated family $\{f^\lambda\}$ of f can be represented by

$$f^\lambda = \Big[\Big(\mathrm{Re}(X^\lambda_{11}) + i\mathrm{Re}(Y^\lambda_{11}), \mathrm{Im}(X^\lambda_{11}) + i\mathrm{Im}(Y^\lambda_{11}), \mathrm{Re}(X^\lambda_{21}) + i\mathrm{Re}(Y^\lambda_{21}), \mathrm{Im}(X^\lambda_{21}) + i\mathrm{Im}(Y^\lambda_{21})\Big)\Big].$$

Corollary 3.7. *Let F_λ be the extended frame defined above. Define a map*

$$\Phi_\lambda = (\phi_\lambda, \psi_\lambda) : \mathbb{D} \to \mathbb{H}^2 \times \mathbb{H}^2 \tag{3.13}$$

by $(\phi_\lambda, \psi_\lambda) := (iF_\lambda\sigma_3 F_\lambda^{-1}, iF_{i\lambda}\sigma_3 F_{i\lambda}^{-1})$. Then $\{\Phi_\lambda\}_{\lambda\in\mathbb{S}^1}$ is a family of spacelike minimal Lagrangian surfaces.

4 Examples through the DPW method

4.1 Basic examples

The basic examples are the open part of the diagonal surface and the product of geodesics, see also in [5].

4.1.1 Open part of the diagonal surface

Define

$$\xi := \lambda^{-1}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\mathrm{d}z$$

for $z \in \mathbb{C}$. It is easy to solve the ODE $\mathrm{d}\Phi = \Phi\xi$ by $\Phi = \exp(z\xi/\mathrm{d}z)$ with $\Phi(0) = \mathrm{Id}$. Moreover, the Iwasawa decomposition of $\Phi = F_\lambda B$ is given by

$$F_\lambda = \frac{1}{\sqrt{1-|z|^2}}\begin{pmatrix}1 & z\lambda^{-1}\\ \bar{z}\lambda & 1\end{pmatrix}.$$

By Proposition 3.6, we obtain a family of open parts of the diagonal surface $\{f^\lambda\}$ parameterized by $\lambda \in \mathbb{S}^1$:

$$f^\lambda = \left[\left(1 - i|z|^2,\ -|z|^2 + i,\ z\lambda^{-1} - i\bar{z}\lambda,\ -\bar{z}\lambda + iz\lambda^{-1}\right)\right].$$

4.1.2 Product of geodesics

Define

$$\xi := \lambda^{-1}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\mathrm{d}z$$

for $z \in \mathbb{C}$. It is easy to solve the ODE $\mathrm{d}\Phi = \Phi\xi$ by $\Phi = \exp(z\xi/\mathrm{d}z)$ with $\Phi(0) = \mathrm{Id}$. Moreover, the Iwasawa decomposition of $\Phi = F_\lambda B$ is given by

$$F_\lambda = \begin{pmatrix}\cosh(\lambda^{-1}z + \bar{z}\lambda) & \sinh(\lambda^{-1}z + \bar{z}\lambda)\\ \sinh(\lambda^{-1}z + \bar{z}\lambda) & \cosh(\lambda^{-1}z + \bar{z}\lambda)\end{pmatrix}.$$

By Proposition 3.6, we obtain a family of products of geodesics $\{f^\lambda\}$ parameterized by $\lambda \in \mathbb{S}^1$:

$$f^\lambda = \left[(\cos s,\ i\cos t,\ i\sin s,\ \sin t)\right],$$

where

$$s = \lambda^{-1}z - \bar{z}\lambda - i(\lambda^{-1}z + \bar{z}\lambda), \quad t = \lambda^{-1}z - \bar{z}\lambda + i(\lambda^{-1}z + \bar{z}\lambda).$$

4.2 Equivariant and radially symmetric examples

We now show two new examples of spacelike minimal Lagrangian surfaces in Q_2^*.

4.2.1 $\mathbb{R}$-equivariant spacelike minimal Lagrangian surfaces

Definition 4.1 ($\mathbb{R}$-equivariant potentials, [2]). *Define*

$$\xi = A(\lambda)\frac{\mathrm{d}z}{z}, \quad \textit{where} \quad A(\lambda) = \begin{pmatrix}c & a\lambda^{-1} + b\lambda\\ -a\lambda - b\lambda^{-1} & -c\end{pmatrix}, \tag{4.1}$$

with $a, b \in \mathbb{R}^*$ and $c \in \mathbb{R}$ for z in $\Sigma = \{z = x + iy \in \mathbb{C} \mid -\kappa_1^2 < x < \kappa_2^2\}$. And we choose κ_1, κ_2 so that $x \in (-\kappa_1^2, \kappa_2^2)$ is the largest interval for which a solution $v = v(x)$ of

$$(v')^2 = (v^2 - 4a^2)(v^2 - 4b^2) + 4c^2v^2, \quad v'' = 2v(v^2 - 2a^2 - 2b^2 + 2c^2), \quad v(0) = 2b,$$

is finite and never zero ($'$ denotes $\frac{\mathrm{d}}{\mathrm{d}x}$). When $c \neq 0$, we require $v'(0)$ and $-bc$ to have the same sign. We call such potentials the equivariant potentials.

It is easy to see that $\Phi = \exp(\log z \cdot A)$ is the unique solution of $\mathrm{d}\Phi = \Phi\xi$ with the initial condition $\Phi(0) = \mathrm{Id}$. Let $\Phi = F_\lambda B$ be the Iwasawa decomposition of Φ (see below for the explicit form of F_λ). As discussed in [2, Section 5.1], by the rotation of the domain $z \to e^{i\theta} \cdot z$, the following transformation rule of F_λ follows:

$$F_\lambda \left(e^{i\theta} \cdot z,\, e^{-i\theta} \cdot \bar{z},\, \lambda\right) = \exp\left(i\theta A(\lambda)\right) \cdot F_\lambda\left(z,\, \bar{z},\, \lambda\right). \tag{4.2}$$

Note that $i\theta A(\lambda)$ takes values in $\Lambda\mathfrak{su}(1,1)_\sigma$ and thus $\exp\left(i\theta A(\lambda)\right)$ takes values in $\Lambda\mathrm{SU}(1,1)_\sigma$.

The general definition of an equivariant surface can be found in [2], then a straightforward computation shows that the spacelike minimal Lagrangian surface constructed by the equivariant potential ξ in (4.1) is an equivariant surface.

Proposition 4.2. *Let ξ be an $\mathbb{R}$-equivariant potential defined in* (4.1) *and let $F_\lambda \in \Lambda\mathrm{SU}(1,1)_\sigma$ for $\lambda \in \mathbb{S}^1$ be the corresponding extended frame. Then the spacelike minimal Lagrangian surface $f^\lambda : M \to Q_2^*$ constructed by $(F_\lambda, F_{i\lambda})$ is equivariant, that is,*

$$\hat{\mathfrak{f}}^\lambda \left(e^{i\theta} \cdot z,\, e^{-i\theta} \cdot \bar{z},\, \lambda\right) = \psi\left(\left(\exp(i\theta A(\lambda)), \exp(-i\theta A(i\lambda))\right) \hat{\mathfrak{f}}^\lambda(z, \bar{z}, \lambda)\right)$$

holds, where $\hat{\mathfrak{f}}^\lambda$ is the horizontal lift of f^λ and $\psi : \Lambda\mathrm{SU}(1,1)_\sigma \times \Lambda\mathrm{SU}(1,1)_\sigma \to \Lambda\mathrm{SO}_0(2,2)_\sigma$ is the loop group homomorphism.

By Theorem 5.1 in [2], we can obtain that

$$F_\lambda = \begin{pmatrix} \sqrt{\frac{4ab\lambda^2+v^2}{2v(a\lambda^2+b)}}\left(\cosh\hat{t} - \frac{c\lambda}{t}\sinh\hat{t}\right) & \frac{-\lambda t(2cv+v')\cosh\hat{t} + (2t^2v+c\lambda^2v')\sinh\hat{t}}{t\sqrt{2v(a\lambda^2+b)(4ab\lambda^2+v^2)}} \\ \frac{\sqrt{(a\lambda^2+b)(4ab\lambda^2+v^2)}}{t\sqrt{2v}}\sinh\hat{t} & \sqrt{\frac{2v(a\lambda^2+b)}{4ab\lambda^2+v^2}}\left(\cosh\hat{t} - \frac{\lambda v'}{2tv}\sinh\hat{t}\right) \end{pmatrix},$$

where $\mathbf{f}$, t and $\hat{t}$ are given by

$$\mathbf{f}(x) = \int_0^x \frac{2\mathrm{d}s}{1 + (4ab\lambda^2)^{-1}v^2(s)}, \quad t = \sqrt{-ab - (a^2+b^2-c^2)\lambda^2 - ab\lambda^4}, \quad \hat{t} = t(\mathbf{f} - z)\lambda^{-1}.$$

It is also easy to compute the map $F_{i\lambda}$. Then by Proposition 3.6, we obtain the explicit form of this equivariant surface in Q_2^*.

4.2.2 Radially symmetric spacelike minimal Lagrangian surfaces

Definition 4.3 (Radially symmetric potentials, [2])**.** *Define*

$$\xi = \lambda^{-1}\begin{pmatrix} 0 & 1 \\ cz^k & 0 \end{pmatrix} dz, \tag{4.3}$$

for $z \in \Sigma = \mathbb{C}$ and $k \in \mathbb{N}$ and some $c \in \mathbb{C} \setminus (\mathbb{S}^1 \cup \{0\})$. Here we call such potentials the radially symmetric potentials.

Let $R_\ell(z) = e^{2\pi i\ell/(k+2)}\bar{z}$ be the reflections of the domain $\mathbb{C}$, for $\ell \in \{0, 1, \ldots, k+1\}$. Note that

$$\xi(R_\ell(z), \lambda) = A_\ell \xi(\bar{z}, \lambda) A_\ell^{-1}, \quad \text{with} \quad A_\ell = \begin{pmatrix} e^{\frac{\pi i\ell}{k+2}} & 0 \\ 0 & e^{\frac{-\pi i\ell}{k+2}} \end{pmatrix} \in \mathrm{SU}(1,1) \tag{4.4}$$

holds. Let Φ be the solution of $d\Phi = \Phi\xi$ with $\Phi(z_0) = \mathrm{Id}$ and consider the Iwasawa decomposition $\Phi = F_\lambda B$. For $c \in \mathbb{C}\setminus(\mathbb{S}^1 \cup \{0\})$, the Iwasawa decomposition of Φ cannot be carried out explicitly, and hence an explicit description of all radially symmetric surfaces cannot be obtained. By (4.4), we have

$$F(R_\ell(z), \overline{R_\ell(z)}, \lambda) = A_\ell F(\bar{z}, z, \lambda) A_\ell^{-1}.$$

This leads to the following proposition.

Proposition 4.4. *Let ξ be the radially symmetric potential defined in* (4.3) *and let $F_\lambda \in \Lambda\mathrm{SU}(1,1)_\sigma$ for $\lambda \in \mathbb{S}^1$ be the extended frame obtained by ξ. The spacelike minimal Lagrangian surface $f^\lambda : M \to Q_2^*$ constructed by $(F_\lambda, F_{i\lambda})$ admits discrete rotational symmetries:*

$$\hat{\mathfrak{f}}^\lambda(R_\ell(z), \overline{R_\ell(z)}, \lambda) = \mathcal{A}_\ell \hat{\mathfrak{f}}^\lambda(\bar{z}, z, \lambda), \quad \textit{with} \quad \mathcal{A}_\ell := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\left(\frac{2\pi\ell}{k+2}\right) & \sin\left(\frac{2\pi\ell}{k+2}\right) \\ 0 & 0 & -\sin\left(\frac{2\pi\ell}{k+2}\right) & \cos\left(\frac{2\pi\ell}{k+2}\right) \end{pmatrix},$$

where $\hat{\mathfrak{f}}^\lambda$ denotes the horizontal lift of f^λ. Moreover, the induced metric of f^λ depends only on the radial coordinate $|z|$. Such a surface f^λ is therefore called radially symmetric.

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